The space of (generalized) Taub-Nut spacetimes*

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Abstract. *In this paper we show how to construct all analytic solutions of the vacuum Einstein equations having a compact Cauchy horizon diffeomorphic to* S^3 *and ruled by closed null generators which fiber the horizon in the sense of Hopf. The set of (inequivalent) solutions is infinite dimensional, contains the two parameter Taub-NUT family as a special case, and may be uniquely parameterized by a pair of arbitrary, real analytic functions on* S^2 *(modulo an action of the conformal group of* S^2*). The horizon of each such solution is necessarily a Killing horizon (as proven recently by Isenberg and the author) and is shown here always to be a «crushing» horizon in the sense of Eardley and Smarr. Some recent results of Gerhardt are used to show that a neighborhood of the horizon (in the globally hyperbolic region) is always foliated by constant mean curvature hypersurfaces.*

The possible isometry groups of the solutions considered are characterized in terms of isometries of the determining «Cauchy data» which is specified on the horizons themselves.

1. INTRODUCTION

The two-parameter, Taub-NUT family of vacuum spactimes has provided important examples of the possible pathological behavior of solutions of Einstein's equations. In Misner's words [1] they give «a counterexample to almost anything» one might naively have conjectured about Einstein spacetimes. The now familiar pathologies exhibited by the Taub-NUT solutions include:

a) the existence of smooth, compact Caucy horizons lying between the globally hyperbolic Taub regions and the causality violating NUT (Newman-Unti-

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-Tamburino) regions,

b) the occurrence of closed timelike curves through every event of the NUT regions,

c) the existence of incomplete geodesics in maximally extended spacetimes which have no curvature singularities, and

d) the possibility of making inequivalent NUT-like extensions of Taub space which however, cannot simulataneously be accommodated within a Hausdorff manifold.

The causality violating feature is particularly disturbing since it suggests a breakdown of the cosmic censorship idea. On the other hand, the Taub-Nut solutions are very special. They each have an $SU(2) \times U(1)$ group of isometries which acts transitively on a preferred family of hypersurfaces diffeomorphic to the three sphere. These hypersurfaces are spacelike in the Taub regions, timelike in the NUT regions and null where they coincide with the Cauchy horizons. It is natural to ask whether one can construct larger families of vacuum spacetimes with the same quantitative Taub-NUT behavior by relaxing the symmetry restrictions upon the metric. The purpose of this paper is to show that indeed one can relax the $SU(2)$ factor in the symmetry group and construct a large family of Taub-NUT-like vacuum spacetimes which each have only a $U(1)$ isometry group. The family we construct is infinite dimensional and has (roughly speaking) half the dimension of the full space of $U(1)$-symmetric, vacuum solutions.

The $SU(2)$ factor in the Taub-NUT symmetry group is the common isometry group of all Bianchi IX («Mixmaster») cosmological models. The $U(1)$ factor however plays a more important role with regard to the extendibility of Taub space through a Cauchy horizon. The Killing field generating this $U(1)$ group action turns null on the Cauchy horizon and becomes tangent to the horizon's null generators. This is no accident. In a recent paper Moncrief and Isenberg [2] proved that any analytic (electro-) vacuum spacetime which contains a comapct null surface ruled (in the sense of an S^1 bundle) by closed null generators, necessarily has a one-dimensional isometry group with a Killing field tangent to the null generators of the surface. Whether one can relax the closure condition on the null generators and find analytic solutions with no continuous symmetries is an open question. Moncrief and Isenberg conjectured that such solutions do not exist.

In this paper we shall consider analytic, vacuum spacetimes which each contain a compact null surface diffeomorphic to S^3. We shall impose the topological constraint that the generators of each such null surface are closed curves which fiber that surface in the sense of Hopf (yielding a non-trivial S^1 bundle: $S^3 \to S^2$). The results of Moncrief and Isenberg [2] show that each such spacetime necessarily has a $U(1)$ isometry group whose action preserves the null generators of the compact null hypersurface. Imposing the implied symmetry upon the metrics considered we prove a local existence theorem for solutions of Einstein's equations of the geometrical type considered. We prove, using a slight extension of the Cauchy-Kowalewski theorem, that any analytic triple $\{\gamma, g_{ab}, \beta_a\}$ defined over S^2 (where γ is a function, g_{ab} is a Riemannian metric and β_a is a one-form) determines an analytic, vacuum spacetime which contains a compact Cauchy horizon diffeomorphic to S^3.

Many of the solutions so constructed are diffeomorphic to one another but one can parametrize the inequivalent solutions in an elegant way. By considering the action of horizon preserving diffeomorphisms which commute with the $U(1)$ symmetry action one can transform the initial data sets $\{\gamma, g_{ab}, \beta_a\}$ to a canonical «gauge». One can first transform g_{ab} (conformally and diffeomorphically) to coincide with the standard (constant curvature) metric on S^2. Without disturbing this condition one can transform β_a so that its divergence relative to g_{ab} is zero. Finally, without disturbing these condizions one can quotient out an action of the conformal group of S^2 on the space of pairs $\{(\gamma, \beta_a) \,|\, \delta_g \beta = 0\}$. The orbits of this group action uniquely parameterize the inequivalent, *extendible*, analytic vacuum spacetimes which we shall call *generalized Taub-NUT spacetimes*.

The techniques for deriving these results were developed previously in Ref. [3] for the special case of a null surface diffeomorphic to T^3 (regarded as a product S^1 bundle: $T^3 \to T^2$). The results of that paper extend immediately to encompass any other *trivial*, compact S^1 bundle. One need only modify the definitions in an obvious way and repeat the arguments of Ref. [3] essentially word for word. The main contribution of the present paper is to show, in a special case, how such arguments can be extended to handle compact, *non-trivial* S^1 bundles. The techniques employed could certainly be applied to treat arbitrary such bundles but, to make the construction as explicit as possible, we have restricted our attention to the Hopf bundle ($S^3 \to S^2$). The analysis is then facilitated by writing out the various geometrical objects of interest in terms of the usual invariant basis vector fields and one-forms defined over S^3.

There is a further generalization of the aforementioned results which one could carry out. Isenberg and the author have recently found [4] that the orbit-closure condition is sufficient to prove their result on the existence of a $U(1)$ isometry group. One can relax the local product requirement upon the structure of the null generators and look for compact null surfaces ruled in the sense of Seifert manifolds [5] (which include S^1 bundles as special cases). Such analytic (electro-) vacuum spacetimes must also admit $U(1)$ isometry groups which preserve the null generators. The picture to keep in mind is that these manifolds may contain exceptional orbits about which the nearby orbits twist in barber pole fashion - closing but not yielding the local product structure of an S^1 bundle. Examples of such spacetimes are provided by the Kerr-Taub-NUT metrics [6] for suitably commensurate values of the adjustable parameters of this family (a restriction needed to ensure the closure of the null orbits). It seems clear that one could extend the arguments of the present paper not only to other non-trivial S^1 bundles but also to Seifert manifolds and thereby prove the existence of infinite dimensional families of still more exotic vacuum spacetimes containing Seifert-Manifold Cauchy horizons. The key idea in such an extension is that one can pass to a suitable covering space which is an S^1 bundle, carry out construction arguments analogous to those given here and then consistently project the resulting vacuum metrics back to the original manifold. This line of argument seems to work in the context of the Moncrief-Isenberg proof of existence of a $U(1)$ symmetry and we believe it should work for the construction of solutions also.

An interesting application of such a construction might be to show that not every analytic vacuum spacetime with a null surface diffeomorphic to S^3 extends to contain a second null surface diffeomorphic to S^3. The Kerr-Taub-NUT solutions do extend to contain such surfaces but, except for the special case of the Taub-NUT solutions, these spacetimes have distinct Killing fields generating their two Cauchy horizons (i.e., there are two commuting $U(1)$ actions which each preserve the generators of only one the two horizons). The above remarks suggest that one could perturb the Kerr-Taub-NUT solutions slightly in such a way as to preserve one of the two $U(1)$ actions and its associtad Cauchy horizon while destroying the symmetry needed for the existence of the second horizon.

While we shall not attempt to carry out such a program here we shall study (within the context of the Hopf bundle structure) the necessary and sufficient conditions upon the null surface initial data for the existence of additional non-trivial, isometries. We show that any such additional isometry must preserve the horizon and commute with the $U(1)$ action implied by the Moncrief-Isenberg theorem. We shall also show how to characterize such additional symmetries in terms of the null surface initial data $\{\gamma, g_{ab}, \beta_a\}$ described above.

2. CONSTRUCTION OF (GENERALIZED) TAUB-NUT SPACETIMES

A. Analytic Lorentz Metrics Over $S^3 \times R$

We wish to consider analytic Lorentzian metrics over $S^3 \times R$ with the property that the resulting spacetimes are foliated by spacelike hypersurfaces diffeomorphic to S^3. We shall then consider the possibility of analytically extending such metrics through compact null surfaces diffeomorphic to S^3.

Let $\{\hat{\omega}^i\} = \{\hat{\omega}^1, \hat{\omega}^2, \hat{\omega}^3\}$ be the standard basis for invariant one-forms defined over S^3. In terms of Euler angle coordinates $\{x^i\} = \{\theta, \phi, \psi\} \in \{[0, \pi), [0, 2\pi), [0, 4\pi)\}$ these basis one-forms may be written

$$\hat{\omega}^1 = \cos\psi \, d\theta + \sin\psi \sin\theta \, d\phi$$

$$\hat{\omega}^2 = -\sin\psi \, d\theta + \cos\psi \sin\theta \, d\phi$$

$$\hat{\omega}^3 = d\psi + \cos\theta \, d\phi. \tag{2.1}$$

They provide a global analytic basis for one-forms over the three-sphere and satisfy the identity

$$\hat{\omega}^i \wedge \hat{\omega}^j = -\epsilon_{ijk} \, d\hat{\omega}^k \tag{2.2}$$

where ϵ_{ijk} is completely antisymmetric and $\epsilon_{123} = 1$. A brief review of the properties of the Euler angles, invariant basis forms, etc. is given in Appendix A.

As discussed in the introduction we only wish to consider those spacetimes which have non-trivial, one-dimensional isometry groups with closed orbits and a local product bundle structure. More precisely, we consider spacetimes with a time function t whose level surfaces are hypersurfaces diffeomorphic to S^3 which are foliated (in the manner of a Hopf fibration) by the orbits of the one-dimensional isometry group. Metrics of this type can always be written (after a change of coordinates to eliminate the «shift vector» ${}^{(4)}g_{0i}$) in the form

$$ds^2 = {}^{(4)}g_{\mu\nu} \, dx^\mu \, dx^\nu = e^{-2\gamma}[-N^2 dt^2 + \hat{g}_{ab} \, \hat{\omega}^a \, \hat{\omega}^b] + t^2 e^{2\gamma} [k\hat{\omega}^3 + \hat{\beta}_a \hat{\omega}^a]^2 \tag{2.3}$$

where $\mu, \nu, \ldots$ range over $(0, 1, 2, 3)$ with $x^0 = t$; $a, b, \ldots$ range over $(1, 2)$ and $i, j, \ldots$ range over $(1, 2, 3)$. The component functions $\gamma, N, \hat{g}_{ab}$ and $\hat{\beta}_a$ are assumed to be analytic on $S^3 \times R$ and to satisfy the conditions that $N > 0$ everywhere and $\hat{g}_{ab}$ is an everywhere positive definite, symmetric 2×2 matrix. These fields are assumed to be chose so that ${}^{(4)}g$ is invariant relative to the one--parameter group generated by $Y = \dfrac{\partial}{\partial\psi}$ (i.e., so that ${}^{(4)}g_{\mu\nu,\psi} = 0$). The integral curves of Y define, in each t = constant hypersurface, a Hopf fibration of S^3. We consider only the regions $t > 0$ or $t < 0$ and require k to be a non-zero constant.

One arrives at this metric form by considering first the general analytic Lorentz metric with t = constant spacelike surfaces diffeomorphic to S^3, imposing the symmetry condition of invariance with respect to $\dfrac{\partial}{\partial\psi}$ and then showing that the spatial coordinate transformation needed to eliminate the shift vectors does not disturb the invariance with respect to $\dfrac{\partial}{\partial\psi}$. The factor of t^2 in Eq. (2.3) could of course be absorbed into the definitions of $e^{2\gamma}$, etc., but has been introduced for later convenience.

After substituting the explicit expressions (2.1) for $\{\hat{\omega}^i\}$ into (2.3) one may reexpress the metric ${}^{(4)}g$ as

$$\begin{aligned} ds^2 = {}^{(4)}g_{\mu\nu}\, dx^\mu\, dx^\nu = e^{-2\gamma}[-N^2 dt^2 + g_{ab}\, dx^a dx^b] \\ + t^2 e^{2\gamma}[k(d\psi + \cos\theta\, d\phi) + \beta_a\, dx^a]^2 \end{aligned} \tag{2.4}$$

where $\{x^a\} = \{\theta, \phi\} \in \{[0, \pi), [0, 2\pi)\}$. These coordinates may be regarded, for each fixed $t =$ constant, as standard polar coordinates over S^2 (the quotient of S^3 by the circle action generated by $\frac{\partial}{\partial\psi}$). For each fixed t, γ and N may be viewed as analytic functions over S^2, $g_{ab}\, dx^a\, dx^b$ may be viewed as an analytic, Riemannian metric and $\beta_a\, dx^a$ may be regarded as an analytic one-form field over S^2. By assumption all such fields depend analytically on t. Conversely, any chosen set of analytic fields $\{\gamma, N, g_{ab}, \beta_a\}$ of this type yields an analytic metric over $S^3 \times R$ of the desired kind. This conclusion follows from comparing the requirements of analyticity and $\frac{\partial}{\partial\psi}$ — invariance of fields of the form $\hat{g}_{ab}\, \hat{\omega}^a \hat{\omega}^b$ and $\hat{\beta}_a \hat{\omega}^a$ defined over S^3 with the requirements of analyticity for the corresponding fields $g_{ab}\, dx^a\, dx^b$ and $\beta_a dx^a$ defined over the quotient manifold $S^2 (\simeq S^3/S^1)$. A brief discussion of this point is given in Appendix A.

The above representation of $\frac{\partial}{\partial\psi}$ — invariant metrics over $S^3 \times R$ resembles that often used in «Kaluza-Klein-Jordan» (KKJ) theories of gravitation. In such KKJ theories one usually deals with trivial S^1 bundles of five dimensions. Here the bundle is non-trivial but only four dimensional. In this way of thinking γ plays the role of the Jordan scalar field, $(-N^2 dt^2 + g_{ab}\, dx^a dx^b)$ (which is a Lorentz metric over $S^2 \times R$), plays the role of the «spacetime metric» and $(k \cos\theta\, d\phi + \beta_a dx^a)$ plays the role of the «electromagnetic vector potential». This last quantity is not a globally defined one-form over $S^2 \times R$ (since $\cos\theta\, d\phi$ is not) but it does represent a globally defined connection on the S^1 bundle $(S^3 \times R) \to (S^2 \times R)$. As a consequence its curvature corresponds to a globally defined two-form over $S^2 \times R$, $(-k \sin\theta\, d\theta \wedge d\phi + \frac{1}{2}(\beta_{a,b} - \beta_{b,a}) dx^b \wedge dx^a + \beta_{a,t}\, dt \wedge dx^a)$. This plays the role of the «electromagnetic field» in the KKJ picture. A more extensive discussion of Kaluza-Klein type theories over non--trivial bundles bas been given by Miller [7].

Now consider the possibility of analytically extending a metric of the above type through the «boundary surface» at $t = 0$. The coordinates used above are singular at $t = 0$ so, following Ref. [3], we introduce new coordinates $\{t', \theta', \phi', \psi'\}$

with the transformation

$$t' = t^2, \qquad \psi' = \psi - \frac{1}{k}\ln t, \qquad \theta' = \theta, \qquad \phi' = \phi \tag{2.5}$$

(so that, in particular, $x^{a'} = x^a$). It is straightforward to show, by writing out the metric explicitly in the new coordinates, that ${}^{(4)}g$ is analytic and Lorentzian on a neighborhood $N = S^3 \times (-\lambda, \lambda)$ of the surface $t' = 0$ in $S^3 \times R$ provided:

(i) $\gamma(t', x^{a'}), N(t', x^{a'}), \beta_a(t', x^{b'})\,dx^{a'}$ and $g_{ab}(t', x^{c'})\,dx^{a'}\,dx^{b'}$ are analytic on N,

(ii) $N > 0$ and g_{ab} is positive definite everywhere on N,

(iii) $\left(\frac{N^2 - e^{4\gamma}}{4t'}\right)$ is analytic on N.

The transformed metric has the form

$$\begin{aligned} ds^2 &= {}^{(4)}g_{\mu'\nu'}\,dx^{\mu'}dx^{\nu'} \\ &= \frac{-e^{-2\gamma}}{4t'}(N^2 - e^{4\gamma})(dt')^2 + e^{-2\gamma}g_{ab}\,dx^{a'}dx^{b'} \\ &\quad + e^{2\gamma}t'[k(d\psi' + \cos\theta' d\phi') + \beta_a\,dx^{a'}]^2 \\ &\quad + e^{2\gamma}dt'[k(d\psi' + \cos\theta' d\phi') + \beta_a\,dx^{a'}]. \end{aligned} \tag{2.6}$$

For such a metric it is easy to show that:

(iv) $t' = 0$ is a null hypersurface with $\frac{\partial}{\partial\psi'}$ tangent to its null generators,

(v) the Killing field $\frac{\partial}{\partial\psi'}$ is spacelike in the region $t' > 0$ but timelike in the region $t' < 0$ – its orbits there being closed timelike curves.

Spacetimes satisfying the conditions (i) - (iii) above are globally hyperbolic in the regions $t' > 0$ (which were covered by the original charts by either $t > 0$ or $t < 0$), have Cauchy horizons diffeomorphic to S^3 at $t' = 0$ and have closed timelike curves through every event in their acausal extensions, $t' < 0$. We shall call such spacetimes *extendible* spacetimes to signify their extendibility through Cauchy horizons. For each such spacetime one can define a second, inequivalent extension by defining a chart

$$t' = t^2, \qquad \psi' = \psi + \frac{1}{k}\ln t \qquad \theta' = \theta, \qquad \phi' = \phi \tag{2.7}$$

instead of (2.5) and proceeding as above. As is well known for the Taub-NUT spacetime, both extensions cannot be simultaneously accommodated within a Hausdorff manifold.

B. Constructing Extendible Solutions of Einstein's Equations

To construct extendible solutions of Einstein's equations of the geometrical type described above we shall follow the procedure developed in Ref. [3] for constructing such solutions over $T^3 \times R$. The global differences in the two problems are reflected in the fact that here $\gamma, \beta_a \, dx^{a'}$ and $g_{ab} \, dx^{a'} dx^{b'}$ are analytic fields defined over $S^2 \times R$ $(\approx (S^3/S^1) \times R)$ whereas there they were analytic fields defined over $T^2 \times R (\approx (T^3/S^1) \times R)$. The Einstein equations for a metric of the form (2.3) were written out explicitly (in a slightly different notation) in Eqs. (2.4) - (2.6) of Ref. [3]. One need only replace the $\beta_a \, dx^a$ in those equations by $\tilde{\beta}_a \, dx^a = \beta_a \, dx^a + k \cos\theta \, d\phi$ which plays the corresponding role here. The fact that $k \cos\theta \, d\phi$ is not a globally smooth one-form over $S^2 \times R$ is irrelevant since only the quantitites $\tilde{\beta}_{a,t} = \beta_{a,t}$ and $d(\tilde{\beta}_a dx^a) = d(\beta_a dx^a) - k \sin\theta \, d\theta \wedge d\phi$, which are smooth fields over $S^2 \times R$, appear in the equations.

To construct analytic extendible solutions we shall apply a variant of the Cauchy-Kowalewski theorem. As in Ref. [3], it will be convenient to construct solutions $(\gamma, \beta_a, g_{ab}, N)(t, x^a)$ from analytic boundary data at $t = 0$ which are analytic on some neighborhood of $t = 0$ in the original chart. It will follow from the Einstein equations that such solutions are even functions of t and thus also analytic functions of $\{t', x^a\}$ on some neighborhood of $t' = 0$. From the analyticity of Einstein's equations it will follow that the analytically continued fields satisfy the Einstein equations in the acausal region $t' \leqslant 0$ which was not covered by the original chart. This procedure of constructing extendible solutions by analytic continuation from solutions in the original $\{t, x^a\}$ coordinates is convenient because of the particular version of the Cauchy-Kowalewski theorem which we employ. A direct construction of solutions in the new coordinates $\{t', x^{a'}\}$ would require a different version of the existence theorem. This complication is traceable to the fact that the Einstein evolution equations, restricted by the symmetry condition of $\dfrac{\partial}{\partial \psi'}$ – invariance, change type from hyperbolic to elliptic across the surface $t' = 0$.

To construct solutions from data given at $t = 0$ we shall impose a coordinate condition to fix the «lapse function» N. We choose N to satisfy

$$N = \frac{e^{2\mathring{\gamma}}}{\sqrt{{}^{(2)}\mathring{g}}} \sqrt{{}^{(2)}g} \tag{2.8}$$

where $\mathring{\gamma} = \gamma(0, x^a)$, ${}^{(2)}\mathring{g} = {}^{(2)}g(0, x^a)$ and ${}^{(2)}g(t, x^a)$ is the determinant of $g_{ab}(t, x^c)$. This ensures the regularity condition (iii) above and also implies $\left(\dfrac{N}{\sqrt{{}^{(2)}g}}\right)_{,t} = 0$ which simplifies the form of Einstein's equations. That no loss of generality results from this restriction of the form of metric (other than a specialization of the coordinate system) was proven, for $T^3 \times R$ by theorem (4) of Ref. [2]. A completely analogous argument can be given for the $S^3 \times R$ problem considered here. To facilitate the study of the regions near $\theta = 0$ and $\theta = \frac{\pi}{2}$, however, one should first make a coordinate transformation of the form

$$\tilde{\psi} = \psi' \pm \phi' , \qquad \tilde{x}^a = x^{a'} , \qquad \tilde{t} = t'. \tag{2.9}$$

These transformations remove the singularities in the «vector potential» at $\theta = 0$ (with the choice $\tilde{\psi} = \psi' + \phi$) and at $\theta = \pi/2$ (with $\tilde{\psi} = \psi' - \phi$) and allow the argument of Ref. [2] to be carried out as before.

The coordinate condition (2.8) eliminates certain singular terms from the Einstein equations (which arose because of the singularity of the $\{t, x^i\}$ coordinates near $t = 0$) but does not eliminate all of the singular terms. However, the variant of the Cauchy-Kowalewski theorem which we shall use allows the construction of solutions even in the presence of the remaining singularities. The proof was sketched in Ref. [3]. For completeness we include here in Appendix B a full statement of the proof.

The extended Cauchy-Kowalewski theorem (a special case of which, for single second order equations, was proven by Fusaro [8]) may be stated as follows. Consider the first order system

$$\frac{\partial u^i}{\partial t} + \frac{k_i u_i}{t} = \sum_{j=1}^{N} \sum_{a=1}^{n} A_{ij}^a(u, t, x^b) \frac{\partial u_j}{\partial x^a} + B_i(u, t, x^b) \tag{2.10}$$

(no sum on i) where the k_i are constants (with, however, $k_i \neq -1, -2, \ldots$, etc.) and where $A_{ij}^a(\;)$ and $B_i(\;)$ are analytic functions of $(u_i, t, x^a - \mathring{x}^a)$ on some neighborhood of the origin. Then we have:

THEOREM (1). *Equation* (2.10) *with the initial condition* $u_i(0, x^a) = 0$ *has a unique analytic solution on a neighborhood of* $(t, x^a) = (0, \mathring{x}^a)$.

Proof. See Appendix B.

One can cast the Einstein evolution equations (Eqs. (2.4) of Ref. [3]) into a form suited to the application of Theorem (1) by following the same procedure employed in Ref. [3].

To do this we let $\mathring{\gamma} = \gamma(0, x^a)$, $\mathring{\beta}_a = \beta_a(0, x^b)$ and $\mathring{g}_{ab} = g_{ab}(0, x^c)$ be specified as arbitrary analytic fields over S^2 (with $\mathring{\beta}_a$ an one-form and $\mathring{g}_{ab}$ a Riemannian metric), fix N according to Eq. (2.8) and define new variables

$$\tilde{\gamma} = \gamma - \mathring{\gamma}, \qquad \tilde{\Delta}_a = \tilde{\gamma}_{,a}, \qquad \tilde{V} = \tilde{\gamma}_{,t} \tag{2.11}$$

with similar expressions defined for β_a and g_{ab}. In terms of the new variables the evolution equations may be expressed in the form of Eq. (2.10) and Theorem (1) may be applied to prove the existence of a unique, local, analytic solution with the initial conditions $\tilde{\gamma} = \tilde{\Delta}_a = \tilde{V} = \ldots = 0$ at $t = 0$.

We thus get a unique, local analytic solution of the evolution equations on neighborhood of $(t, x^a) = (0, \mathring{x}^a)$ for any point $\mathring{x}^a$ of S^2. Since each such local solution has a non-zero radius of convergenge and since S^2 is compact we can restrict the set of all such local solutions (for fixed $\mathring{\gamma}$, etc.) to a finite subset which overlap to cover a neighborhood of $t = 0$. The overlapping solutions always coincide on their regions of overlap because of the uniqueness result in Theorem (1).

To prove that the solutions so constructed are even in t (and thus analytic in t' on a neighborhood of $t' = 0$) one shows by successive differentiations of the equations of motion that all the odd time derivatives of $(\gamma, \beta_a, g_{ab})$ vanish at $t = 0$.

To complete the proof of existence of analytic, extendible vacuum solutions with arbitrary initial data $(\mathring{\gamma}, \mathring{\beta}_a, \mathring{g}_{ab})$ we need only show that the constraint equations are also satisfied on a neighborhood of $t' = 0$. However, the constraint functions H and H_i (see Eq. (2.6) and the definitions preceeding Eq. (2.14) of Ref. [3]) vanish identically at $t = 0$ for any solution of the type described above. It then follows from another application of Theorem (1) (to Eq. (2.14) of Ref. [3]) that H and H_i vanish on a neighborhood of $t = 0$ (the argument given in Ref. [3] was slightly more complicated than necessary). Finally, it follows from the analyticity of Einstein's equations in the extendible coordinates $\{t', x^{i'}\}$ and from the analyticity of the constructed solutions that Einstein's full vacuum field equations are satisfied on a neighborhood of $t' = 0$ for arbitrary initial data $\{\mathring{\gamma}, \mathring{\beta}_a, \mathring{g}_{ab}\}$ defined over S^2.

We have thus proven the existence theorem:

THEOREM (2). *Any analytic data $(\gamma, \beta_a, g_{ab})(0, x^c)$ specified over S^2 (with β_a a one-form and g_{ab} a Riemannian metric) determines a unique, analytic, extendible solution of the vacuum Einstein equations on some neighborhood of the (compact, null) initial data hypersurface. In coordinates $\{t', \psi', x^{a'}\}$ adapted to the extension, the solution covers a neighborhood of a smooth Cauchy horizon (diffeomorphic to S^3) at $t' = 0$ and has $\frac{\partial}{\partial \psi'}$ as a globally defined Killing vector field. This Killing field is spacelike in the globally hyperbolic region $t' > 0$, null on the horizon and timelike in the acausal extension $t' < 0$; its orbits determine* *a Hopf fibration of each t' = constant hypersurface.*

It is worth noting the Cauchy horizons determined by the above construction are always «crushing singularities» in the sense of Eardley and Smarr [9]. In other words, the trace of the second fundamental form, $\mathrm{tr}(K_t)$, induced on the t = constant surfaces blows up uniformly as $t \to 0^+$. To see this one need only write out the explicit expression for $\mathrm{tr}(K_t)$,

$$\mathrm{tr}(K_t) = -\frac{e^{\gamma}}{N}\left\{\frac{1}{t} - \gamma_{,t} + \frac{1}{2} g^{ab} g_{ab,t}\right\}. \tag{2.12}$$

Since γ, g_{ab} and N are analytic on a neighborhood of $t = 0$, it is easy to see from Eq. (2.12) that there exist constants $t_0 > 0$, $t_1 > 0$, $\kappa_0 > 0$, $\kappa_1 > 0$ with $t_1 < t_0$ and $\kappa_1 > \kappa_0$ such that

$$(-\mathrm{tr}(K_{t_0})) < \kappa_0, \qquad (-\mathrm{tr}(K_{t_1})) > \kappa_1 \tag{2.13}$$

It follows from some recent work of Gerhardt [10] that there exists a smooth spacelike constant mean curvature hypersurface Σ lying between the hypersurfaces Σ_{t_0} and Σ_{t_1} and having $\kappa_0 < (-\mathrm{tr}(K_\Sigma)) < \kappa_1$. Since the spacetime bounded by Σ and the «crushing singularity» at $t' = 0$ is globally hyperbolic, non-singular and «crushing» it also follows from Gerhardt's work that this spacetime is foliated by smooth, spacelike, constant-mean-curvature hypersurfaces whose mean curvature varies monotonically from $-\kappa = \mathrm{tr}(K_\Sigma)$ to $-\infty$. An earlier result of this type, for the special case of Gowdy spacetimes on $T^3 \times R$, was obtained by Isenberg and Moncrief [11]. In that case one could prove the existence of global foliations for the maximal Cauchy developments since those developments had already been characterized by a global existence argument of Moncrief [12].

It is straightforward to show that the two parameter family of Taub-NUT solutions emerge as special cases of our general construction. These solutions arise from the initial conditions $\mathring{\beta}_a = 0$, $\mathring{\gamma}$ = constant, and $\mathring{g}_{ab}\,\mathrm{d}x^a \mathrm{d}x^b = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

The adjustable constants k (see Eq. (2.3)) and $\mathring{\gamma}$ provide the two Taub-NUT parameters. Our coordinates (t', ψ') are not identical to those ordinarily defined in studying the Taub-NUT solutions. They are, however, analytically related to those coordinates on a neighborhood of the horizon at $t' = 0$.

3. THE SPACE OF (GENERALIZED) TAUB-NUT SPACETIMES

Many of the spacetimes generated according to theorem (2) are of course merely diffeomorphic copies of one another. To characterize the space of inequivalent generalized Taub-NUT spacetimes it is convenient to transform the metrics to a canonical gauge. For this purpose we consider, for each such spacetime, the group of analytic diffeomorphisms which (i) preserve the horizon at $t' = 0$, and (ii) commute with the isometry generated by $\frac{\partial}{\partial\psi'}$. Infinitesimal generators of such diffeomorphisms are vector fields ${}^{(4)}X$ which are (i') tangent to the null hypersurface at $t' = 0$ and (ii') satisfy the invariance condition $\left[\frac{\partial}{\partial\psi'}, {}^{(4)}X\right] = 0$.

In terms of the analytic basis fields (see Appendix A)

$$
\begin{aligned}
\xi_0 &= \frac{\partial}{\partial t'} = \frac{1}{2t}\frac{\partial}{\partial t} + \frac{1}{2kt^2}\frac{\partial}{\partial\psi} \\
\xi_1 &= \frac{\partial}{\partial\phi} \\
\xi_2 &= \cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi} + \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi} \\
\xi_3 &= -\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi} + \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi}
\end{aligned}
\tag{3.1}
$$

any such generator may be expressed as

$$
{}^{(4)}X = t'\hat{Y}\xi_0 + \hat{X}^i\xi_i \tag{3.2}
$$

where $\hat{Y}$ and $\hat{X}^i$ are analytic functions of t' and $x^{a'}$ (independent of $\psi' = x^{3'}$). The infinitesimal diffeomorphism of ${}^{(4)}g$ generated by ${}^{(4)}X$ is of course given by $\delta\,{}^{(4)}g = L_{{}^{(4)}X}\,{}^{(4)}g$ and induces the following infinitesimal transformations of the metric functions γ, β_a, g_{ab}:

$$
\begin{aligned}
\delta\gamma &= \frac{\hat{Y}}{2}(1 + t\gamma_{,t}) + L_{{}^{(2)}X}\gamma \\
\delta\beta_a &= \left(\frac{\hat{Y}}{2}\right)_{,a} + \left(\hat{Y}\frac{t}{2}\right)\beta_{a,t} + \left(L_{{}^{(2)}X}\beta\right)_a \\
&\quad + \hat{X}^1_{,a}k\cos\theta + \hat{X}^2_{,a}k\sin\theta\sin\phi + \hat{X}^3_{,a}k\sin\theta\cos\phi \\
\delta g_{ab} &= \hat{Y}g_{ab} + \left(\hat{Y}\frac{t}{2}\right)g_{ab,t} + \left(L_{{}^{(2)}X}g\right)_{ab}
\end{aligned}
\tag{3.3}
$$

where

$$ {}^{(2)}X = \hat{X}^i(\xi_i)^a \frac{\partial}{\partial x^a} = \hat{X}^i L_i \tag{3.4} $$

We have introduced the notation L_i for $(\xi_i)^a \frac{\partial}{\partial x^a}$ (the vector field ξ_i with the $\frac{\partial}{\partial \psi}$ term discarded) since these objects may be identified with the usual analytic «angular momentum» generators defined over S^2 and, in particular, satisfy

$$ [L_i, L_j] = \xi_{ijk} L_k . \tag{3.5} $$

Thus ${}^{(2)}X$ may be identified with an analytic vector field over $S^2 \times R$. The three functions $\hat{X}^i$ may be recovered from their «vector part» ${}^{(2)}X$ together with their «scalar part» $\hat{X}^i n_i$ where

$$ (n_i) = (\cos\theta, \sin\theta \sin\phi, \sin\theta\cos\phi) \tag{3.6} $$

Restricted to the surface $t' = 0$ the infinitesimal transformations (3.3) reduce to

$$ \begin{aligned} \delta\mathring{\gamma} &= \left\{ \frac{\hat{Y}}{2} + L_{{}^{(2)}X}\gamma \right\} \Big|_{t=0}, \\ \delta\mathring{\beta}_a &= \left\{ \left(\frac{\hat{Y}}{2}\right)_{,a} + (L_{{}^{(2)}X}\beta)_a + k\,\hat{X}^i_{,a} n_i \right\} \Big|_{t=0}, \\ \delta\mathring{g}_{ab} &= \{ \hat{Y} g_{ab} + (L_{{}^{(2)}X} g)_{ab} \} \big|_{t=0} \end{aligned} \tag{3.7} $$

The group property of such transformations of the initial data is reflected in the commutator of such transformations. If $(\hat{Y}, \hat{X}^i)$ and $(\hat{Y}^*, \hat{X}^{i*})$ are the generators of any two such infinitesimal diffeomorphisms then their commutator is a transformation of the same type with a generator $(\hat{Y}', \hat{X}^{i'})$ given by:

$$ \begin{aligned} \hat{Y}' &= L_{{}^{(2)}X^*}\hat{Y} - L_{{}^{(2)}X}\hat{Y}^*, \\ \hat{X}^{i'} &= L_{{}^{(2)}X^*}\hat{X}^i - L_{{}^{(2)}X}\hat{X}^{i*} + \epsilon_{ijk}\hat{X}^{j*}\hat{X}^k \end{aligned} \tag{3.8} $$

These imply that

$$ \begin{aligned} {}^{(2)}X' &= [{}^{(2)}X^*, {}^{(2)}X], \\ \hat{X}^{i'} n_i &= L_{{}^{(2)}X^*}(\hat{X}^i n_i) - L_{{}^{(2)}X}(\hat{X}^{i*} n_i) + n_i \epsilon_{ijk}\hat{X}^j\hat{X}^{k*}. \end{aligned} \tag{3.9} $$

To bring the initial data $(\mathring{\gamma}, \mathring{\beta}_a, \mathring{g}_{ab})$ to a canonical gauge we first apply an element of the abelian subgroup generated by $(\hat{Y}, 0)$. This subgroup acts on

$\mathring{g}_{ab}$ by conformal transformation and can be used to bring $\mathring{g}_{ab}$ to a convenient representative of its conformal equivalence class. For S^2 there is only one conformal equivalence class and a convenient representative of it is the constant curvature metric with area 4π (i.e., the standard metric for S^2).

Next we consider the non-abelian subgroup generated by elements of the form $(0, \hat{X}^j)$ which acts on $\mathring{g}_{ab}$ by diffeomorphisms generated by ${}^{(2)}X$, the vector part of $(\hat{X}^j)$. With this group action we can bring $\mathring{g}_{ab}$ to the canonical form

$$\mathring{g}_{ab}\, dx^a\, dx^b = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{3.10}$$

Next we consider the abelian subgroup generated by elements of the form $(\hat{Y}, \hat{X}^i) = (0, Cn_i)$ where C is an arbitrary analytic function and (n_i) is given in Eq. (3.6). These generators have $\hat{Y} = {}^{(2)}X = 0$ and thus leave $\mathring{\gamma}$ and $\mathring{g}_{ab}$ invariant while transforming $\mathring{\beta}_a$ according to

$$\delta\mathring{\beta}_a = kC_{,a}. \tag{3.11}$$

With C appropriately chosen, one can use this subgroup to transform the divergence of $\mathring{\beta}_a$ to zero.

Finally, without disturbing the foregoing gauge conditions, one can apply the six parameter subgroup (isomorphic to the conformal group of S^2) generated by elements $(\hat{Y}, \hat{X}^i)$ with

$$\begin{aligned} \hat{Y} &= -{}^{(2)}X^a|_a \\ \hat{X}^i &= \ell_i + \epsilon_{ijk}\, c_j n_k - \left(\frac{c_j n_j}{k}\right) n_i \end{aligned} \tag{3.12}$$

where c_i and ℓ_i are arbitrary constants. This corresponds to ${}^{(2)}X$ being an arbitrary conformal Killing field of $\mathring{g}_{ab}\, dx^a\, dx^b = d\theta^2 + \sin^2\theta\, d\phi^2$,

$${}^{(2)}X = \ell_i L_i + c_i \epsilon_{ijk} n_j L_k \tag{3.13}$$

with $\hat{Y}$ chosen to yield $\delta\mathring{g}_{ab} = 0$ and $\hat{X}^i n_i$ chosen to yield $\delta({}^{(2)}\nabla_a \mathring{\beta}^a) = 0$. The commutator of any two such transformations, generated by (ℓ_i, c_i) and (ℓ_i^*, c_i^*), is a transformation of the same type with a generator (ℓ_i', c_i') given by

$$\begin{aligned} \ell_i' &= \epsilon_{ijk}(\ell_j^* \ell_k - c_j^* c_k) \\ c_i' &= \epsilon_{ijk}(c_j^* \ell_k - c_j \ell_k^*). \end{aligned} \tag{3.14}$$

This commutation relation coincides with the Lie algebra of the conformal group of S^2 which is turn coincides with the Lie algebra of the Lorentz group.

Let Conf (S^2) designate this action of the conformal group of S^2 on the data $\{\mathring{\gamma}, \mathring{\beta}_a\}$ where $\mathring{g}_{ab}$ is fixed and given by (3.10), where $\mathring{\beta}_a$ has vanishing divergence with respect to $\mathring{g}_{ab}$ but is otherwise arbitrary and where $\mathring{\gamma}$ is arbitrary. It is natural to identify the inequivalent generalized Taub-NUT spacetimes with the orbits of this group action. We, therefore, define the space S of generalized Taub-NUT spacetimes by (1)

$$S = \frac{\{(\mathring{\gamma}, \mathring{\beta}_a) \mid {}^{(2)}\nabla_a \mathring{\beta}^a = 0\}}{\mathrm{Conf}\,(S^2)} . \tag{3.15}$$

Roughly speaking, this space is parameterized by two arbitrary functions on S^2 and has half the dimension of the full space of $\frac{\partial}{\partial \psi'}$ -symmetric vacuum spacetimes on $S^3 \times R$. Presumably the other symmetric, analytic solutions (i.e., those not represented in S) develop curvature singularities rather than Cauchy horizons at the boundaries of their maximal globally hyperbolic extensions.

In the foregoing we have considered diffeomorphisms which preserve the explicit symmetry $\left(\text{invariance with respect to } \frac{\partial}{\partial \psi'}\right)$ and the initial data surface (the horizon at $t' = 0$) but we have not attempted to preserve the coordinate conditions such as the zero shift condition which we imposed initially to simplify the metric form. Such a restriction was not necessary for the classification of inequivalent initial data sets discussed above. It will be useful to impose this restriction, however, for the discussion of possible additional Killing symmetries to be given below. It is not difficult to show that an infinitesimal generating vector field ${}^{(4)}X$ must be constrained to satisfy the propagation equations

$$\begin{aligned} \hat{Y}_{,t} &= \frac{2N}{\sqrt{{}^{(2)}g}} \frac{\partial}{\partial x^a} \left[\frac{\sqrt{{}^{(2)}g}}{N} \left(\frac{{}^{(2)}X^a - {}^{(2)}\mathring{X}^a}{t} \right) \right] \\ {}^{(2)}X^a_{\ ,t} &= N^2\, t/2\, g^{ab}\, \hat{Y}_{,b} \\ k(n_i \hat{X}^i)_{,t} &= -\left(\frac{\hat{Y}}{2} \right)_{,t} - \beta_a^{(2)} X^a_{\ ,t} \end{aligned} \tag{3.16}$$

in order to preserve the zero shift condition (c.f., the metric form in Eq. (2.3)) and the condition (2.8) upon the lapse function N. The existence theory for solutions of these equations is identical to that discussed for the corresponding equations (3.4) of Ref. [3] and so need not be repeated here. This theory (which

---

(1) We have here suppressed the non-zero constant k which was introduced in the metric form (2.3). This constant provides one additional parameter in the space of generalized Taub-NUT spacetimes.

follows immediately from theorem (1) above) establishes the existence of a unique analytic vector field ${}^{(4)}X$ satisfying Eqs. (3.16) for arbitrary, analytic initial data $(\hat{Y}, \hat{X}^i)\,|_{t=0}$. We shall show below that if the initial data $(\mathring{\gamma}, \mathring{\beta}_a, \mathring{g}_{ab})$ is left invariant by this initial data for ${}^{(4)}X$ then the ${}^{(4)}X$ constructed by the above procedure will be a Killing field of ${}^{(4)}g$. Furthermore, all analytic Killing fields of ${}^{(4)}g$ arise in this way.

We first show that any analytic Killing field of ${}^{(4)}g$ must (i) be tangent to the surface $t' = 0$ and (ii) commute with $\frac{\partial}{\partial\psi'}$. To establish (i) we consider the metric ${}^{(4)}g$ in the primed coordinate system (2.6) and construct the Killing form $K_{\mu'\nu'} = (L_{{}^{(4)}X}\,{}^{(4)}g)_{\mu'\nu'}$ for an arbitrary vector field ${}^{(4)}X$. Setting $K_{3'3'}\,|_{t'=0} =$ $= 0$ leads to the equation

$$(kX^{t'} + X^{t'}_{,3'})\,|_{t'=0} = 0. \tag{3.17}$$

Integrating this and demanding smoothness along the (closed) orbits of $\frac{\partial}{\partial\psi'}$ leads to $X^{t'}\,|_{t'=0} = 0$ which proves (i). In a similar way we get from $K_{3'a'}\,|_{t'=0} = 0$ that $X^{b'}_{,3'}\,|_{t'=0} = 0$ and from $K_{3'3',t'}\,|_{t'=0} = 0$ and $K_{t'3'}\,|_{t'=0} = 0$ equations that imply $X^{3'}_{,3'}\,|_{t'=0} = 0$ and thus $\left[\frac{\partial}{\partial x^{3'}}, {}^{(4)}X\right]\Big|_{t'=0} = 0$. However, if ${}^{(4)}X$ and $\frac{\partial}{\partial x^{3'}} = \frac{\partial}{\partial\psi'}$ are both analytic Killing fields, then so is ${}^{(4)}Z = \left[\frac{\partial}{\partial\psi'}, {}^{(4)}X\right]$. However, ${}^{(4)}Z$ vanishes on the three-dimensional surface $t' = 0$ and thus everywhere. This follows from noting that, using Killing's equations in the analytic case, all the successive t' = derivatives of ${}^{(4)}Z$ can be expressed at $t' = 0$ in terms of the trivial data ${}^{(4)}Z\,|_{t'=0} = 0$. This proves (ii) and thus shows that any analytic Killing field of ${}^{(4)}g$ must be expressible in the form (3.2) discussed above.

Any such vector field induces an infinitesimal gauge transformation of the metric functions γ, β_a, g_{ab} given by Eq. (3.3). Restricted to the initial data surface $t' = 0$ this transformation reduces to that given in Eq. (3.7). Therefore, a necessary condition on the vector field ${}^{(4)}X$ for it to be a Killing field is that it leave the initial data $(\mathring{\gamma}, \mathring{\beta}_a, \mathring{g}_{ab})$ invariant, i.e., that it yield $\delta\mathring{\gamma} = \delta\mathring{\beta}_a = \delta\mathring{g}_{ab} = 0$ in Eq. (3.7). Given «initial data» $(\hat{Y}, \hat{X}^i)\,|_{t=0}$ with this property we construct a unique analytic vector field ${}^{(4)}X$ by solving Eq. (3.16) with these initial conditions. Our aim is to show that the Killing form of this vector field, ${}^{(4)}h =$ $= (L_{{}^{(4)}X}\,{}^{(4)}g)$, vanishes identically.

First of all, ${}^{(4)}h = (L_{{}^{(4)}X}\,{}^{(4)}g)$ satisfies the linearized Einstein equations since, of course, any infinitesimal gauge transformation has this property. Secondly, ${}^{(4)}h$ satisfies the linearized versions of the coordinate conditions initially imposed on ${}^{(4)}g$ since these were precisely equivalent to requiring that ${}^{(4)}X$ satisfy Eqs. (3.16). Therefore $(\delta\gamma, \delta\beta_a, \delta g_{ab})$ satisfy the linearized versions of Eqs. (2.4) of Ref. [3]. These equations are linear and homogeneous in the perturbation variables $(\delta\gamma, \delta\beta_a, \delta g_{ab})$ and have only the trivial solution for the case of vanishing initial data (as follows from an application of Theorem (1) above). It follows that $\delta\gamma = \delta\beta_a = \delta g_{ab} = 0$ everywhere and therefore also that $\delta N = \delta\left(\frac{e^{2\mathring{\gamma}}}{\sqrt{{}^{(2)}\mathring{g}}}\sqrt{{}^{(2)}g}\right) = 0$ since $\delta\mathring{\gamma} = \delta\mathring{g}_{ab} = 0$ by assumption. Thus ${}^{(4)}h = L_{{}^{(4)}X}\,{}^{(4)}g = 0$ and ${}^{(4)}X$ is a Killing field of ${}^{(4)}g$. We thus have:

THEOREM (3). *A necessary and sufficient condition for ${}^{(4)}g$ to admit an analytic Killing field ${}^{(4)}X$ is that initial data $(\hat{Y}, \hat{X}^i)|_{t=0}$ for this vector field exist which leave the initial data for ${}^{(4)}g$ invariant (i.e., which give $\delta\mathring{\gamma} = \delta\mathring{\beta}_a = \delta\mathring{g}_{ab} = 0$ in Eq.* (3.7)). *Any such vector field is tangent to the initial surface $t' = 0$, commutes with the Killing field $\frac{\partial}{\partial\psi'}$ and may be uniquely constructed from its initial data by integrating Eqs.* (3.16).

It's clear from Eqs. (3.7) that, for ${}^{(4)}X$ to be a Killing field of ${}^{(4)}g$, ${}^{(2)}X|_{t=0}$ must be a conformal Killing field of $\mathring{g}_{ab}$ and $\hat{Y}|_{t=0}$ must equal minus the divergence of ${}^{(2)}X|_{t=0}$. If ${}^{(2)}X|_{t=0}$ is actually a *Killing* field of $\mathring{g}_{ab}$ then $\hat{Y}|_{t=0} = 0$ and the unique solution of Eqs. (3.16) is given by

$$\hat{Y} = 0\ ,\quad \hat{X}^i = \hat{X}^i|_{t=0}. \tag{3.18}$$

In this case ${}^{(4)}X$ is time independent in the given coordinate system.

The study of additional Killing symmetries is simplified somewhat if we work in the canonical gauge defined above in which $\mathring{g}_{ab}$ has the form (3.10) and $\mathring{\beta}_a$ has vanishing divergence. In this gauge an independent Killing field (i.e., one independent of $\frac{\partial}{\partial\psi'}$) exists if and only if there exists a generator $(Y, X^i)|_{t=0}$ of the form (3.12) which leaves the initial data $(\mathring{\gamma}, \mathring{\beta}_a, \mathring{g}_{ab})$ invariant. In other words an additional Killing field exists if and only if the initial data is left fixed by a non-trivial subgroup of the conformal group Conf (S^2) discussed above. In defining the space S of inequivalent (generalized) Taub-NUT spacetimes (c.f., Eq. (3.15)) we took a quotient by this group action. We now see that this group acts freely on the space of (canonically gauged) initial data except at those points which have non-trivial additional Killing symmetries.

We can simplify the parameterization of S if we recall that any divergence--free on form $\mathring{\beta}_a$ on S^2 can be expressed as

$$\mathring{\beta}_a = k\mathring{g}_{ab} \frac{\epsilon^{bc}}{\sqrt{{}^{(2)}\mathring{g}}} \mathring{\lambda}_{,c} \tag{3.19}$$

where $\epsilon^{ab} = -\epsilon^{ba}$, $\epsilon^{12} = 1$ and where $\mathring{\lambda}$ is a function on S^2 determined uniquely up to an additive constant. We can remove the arbitrariness in the choice of $\mathring{\lambda}$ by imposing the normalization condition

$$\int_{S^2} \sqrt{{}^{(2)}\mathring{g}}\, \mathring{\lambda} = 0. \tag{3.20}$$

The action of the group Conf (S^2) on $\mathring{\lambda}$ can be readily computed and yields the infinitesimal transformation

$$\delta\mathring{\lambda} = L_{{}^{(2)}X}\mathring{\lambda} - \frac{1}{2}\, {}^{(2)}X^a_{\ |a} + \frac{1}{4\pi} \int_{S^2} \sqrt{{}^{(2)}\mathring{g}}\, ({}^{(2)}X^a_{\ |a}\mathring{\lambda}). \tag{3.21}$$

Aside from the final (spatially constant) term (which is needed to preserve the normalization condition (3.20)) this transformation has the same form as that for $\mathring{\gamma}$ (c.f., Eqs. (3.7) and (3.12)). We can, therefore reexpress S as (2)

$$S = \frac{\left\{(\mathring{\gamma}, \mathring{\lambda}) \mid \int_{S^2} \sqrt{{}^{(2)}\mathring{g}}\, \mathring{\lambda} = 0\right\}}{\text{Conf}\,(S^2)} \tag{3.22}$$

and again characterize the occurrence of additional Killing symmetries in terms of fixed points of the conformal group action.

APPENDIX A

The three sphere $S^3 \equiv \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 1\}$ may be parametrized by Euler angles $(\theta, \phi, \psi) \in \{[0, \pi), [0, 2\pi), [0, 4\pi)\}$ via

$$\begin{aligned} x - iy &= \sin\left(\frac{\theta}{2}\right) \exp\left(i\left(\frac{\psi - \phi}{2}\right)\right) \\ w + iz &= \cos\left(\frac{\theta}{2}\right) \exp\left(i\left(\frac{\psi + \phi}{2}\right)\right). \end{aligned} \tag{A.1}$$

---

(2) Where again we have suppressed the constant k (c.f., Eg. (2.3)) which prevides one additional parameter in the space of solutions.

This is equivalent to expressing an arbitrary element U of $SU(2) \approx S^3$ in the forms

(A.2) $$U = \begin{pmatrix} w + iz & y + ix \\ -y + ix & w - iz \end{pmatrix} = e^{i(\sigma_3 \psi/2)} e^{i(\sigma_1 \theta/2)} e^{i(\sigma_3 \phi/2)}$$

where $\{\sigma_i\}$ are the Pauli matrices.

The vector fields $\{\xi_i\}$ defined by Eq. (3.1) provide an analytic basis for vectors on S^3. Their analyticity may most easily be seen by rexpressing them in terms of $\{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 1\}$, which determine the usual analytic structure on S^3. The one-forms $\{\hat{\omega}^i\}$ defined by Eq. (2.1) provide a corresponding analytic basis for one-forms on S^3 and are invariant with respect to the $\{\xi_i\}$ (i.e., $L_{\xi_i} \hat{\omega}^j = 0$). The explicit forms of these basis fields and their respective dual bases are given, in both types of coordinates, by Miller in Ref. [6].

The orbits of $\dfrac{\partial}{\partial \psi}$ (a vector field in the basis dual to the $\{\hat{\omega}^i\}$) define the fibers of S^3, regarded as a non-trivial S^1 bundle over S^2. One may think of $(\theta, \phi) \in \{[0, \pi), [0, 2\pi)\}$ as parameterizing (as standard spherical coordinates) the orbit space S^2. An analytic projection map $\pi : S^3 \to S^2$ is defined which sends any point in S^3 to the point labeling the orbit on which it lies in S^2.

Any analytic function on S^2 may be lifted (i.e., pulled back using π) to an analytic function on S^3 which is invariant under the flow generated by $\dfrac{\partial}{\partial \psi}$ and, conversely, any analytic, $\dfrac{\partial}{\partial \psi}$ – invariant function on S^3 projects to an analytic function on S^2.

A similar remark holds for one forms on S^2 but one must keep in mind that any smooth one form vanishes somewhere on S^2 and thus that a global basis does not exist. One can cover S^2 by using, in appropriate regions, the analytic (local) bases $(d\bar{x}, d\bar{y})$, $(d\bar{x}, d\bar{z})$ and $(d\bar{y}, d\bar{z})$ where $\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = 1$. Adopting the usual parameterization of S^2:

(A.3) $$\begin{aligned} \bar{x} &= \sin\theta \cos\phi \\ \bar{y} &= \sin\theta \sin\phi \\ \bar{z} &= \cos\theta \end{aligned}$$

we get (locally defined) bases $(d(\sin\theta \cos\phi), d(\sin\theta \sin\phi))$, etc., expressed in spherical coordinates. Pulling these basis forms back to S^3 we get analytic, $\dfrac{\partial}{\partial \psi}$ – invariant forms expressible as

$$\begin{aligned} d(\sin\theta\cos\phi) &= (w^2 - z^2 - x^2 + y^2)\,\hat{\omega}^1 - 2(wz + yx)\,\hat{\omega}^2, \\ d(\sin\theta\sin\phi) &= 2(wz - yx)\,\hat{\omega}^1 + (w^2 - z^2 + x^2 - y^2)\,\hat{\omega}^2, \\ d(\cos\theta) &= -2(wx + yz)\,\hat{\omega}^1 + 2(xz - wy)\,\hat{\omega}^2. \end{aligned} \tag{A.4}$$

It follows that the pull back of any analytic one form β on S^2 is expressible as a $\frac{\partial}{\partial\psi}$ – invariant form $\hat{\beta}_a\hat{\omega}^a$ on S^3. The local bases for such forms on S^2 pull back to corresponding local bases on S^3. Conversely, any analytic $\frac{\partial}{\partial\psi}$ – invariant form on S^3 which annihilates $\frac{\partial}{\partial\psi}$ (i.e., which is expressible purely in terms of the $\{\hat{\omega}^a\}$) projects to an analytic form on S^2.

Finally, by the same reasoning, any analytic Riemannian metric on S^2 pulls back to a symmetric, analytic $\frac{\partial}{\partial\psi}$ – invariant tensor on S^3 expressible as $\hat{g}_{ab}\hat{\omega}^a\hat{\omega}^b$. Conversely any analytic, symmetric $\frac{\partial}{\partial\psi}$ – invariant two-tensor on S^3 which annihilates $\frac{\partial}{\partial\psi}$ and is positive definite on subspace complementary to $\frac{\partial}{\partial\psi}$ projects to an analytic Riemannian metric on S^2.

APPENDIX B

We prove here the extended Cauchy-Kowalewski theorem discussed in Section II. A special case (for single second-order equations) was proven by Fusaro in Ref. [8]. Consider a system of partial differential equations expressible in the form

$$\frac{\partial u_i}{\partial t} + \frac{k_i u_i}{t} = \sum_{j=1}^{N}\sum_{a=1}^{n} A^a_{ij}(u, x, t)\,\frac{\partial u_j}{\partial x^a} + B_i(u, x, t) \tag{B.1}$$

(no sum on i) where $A^a_{ij}(\ \ )$ and $B_i(\ \ )$ are (real) analytic functions of $(u, x, t) = \{u_i, x^a, t\}$ on a neighborhood of the origin and where the k_i are (real) constants (with, however, $k_i \neq -1, -2, \ldots$, etc.). We wish to solve this equation subject to the initial condition $u_i(x, 0) = 0$. We shall do this by constructing the formal series expression for the solution and by proving its convergence by comparison with the companion, non-singular problem

$$\text{(B.2)} \qquad \frac{\partial u_i}{\partial t} = K\left\{\sum_{j=1}^{N}\sum_{a=1}^{n} A_{ij}^{a}(u,x,t)\,\frac{\partial u_j}{\partial x^a} + B_i(u,x,t)\right\},$$

$$u_i(x,0) = 0$$

where K is a suitably chosen positive constant.

Using the multi-index notation of Schwartz [13] we write

$$\text{(B.3)} \qquad \begin{aligned} A_{ij}^{a}(u,x,t) &= \Sigma A_{ij}^{a\{\alpha\beta\lambda\}}(u_1^{\alpha_1}u_2^{\alpha_2}\dots u_N^{\alpha_N}\times(x^1)^{\beta_1}\dots(x^n)^{\beta_n}t^{\lambda}) \\ &= \Sigma A_{ij}^{a\{\alpha\beta\lambda\}}u^{\alpha}x^{\beta}t^{\lambda}, \\ B_i(u,x,t) &= \Sigma B_i^{\{\alpha\beta\lambda\}}u^{\alpha}x^{\beta}t^{\lambda} \end{aligned}$$

and seek a solution of the form

$$\text{(B.4)} \qquad u_i = \sum_{\{\beta\}}\sum_{\lambda=1}^{\infty} u_i^{\{\beta\lambda\}}x^{\beta}t^{\lambda}$$

(where we set $u_i^{\{\beta,0\}} = 0$ to implement the initial condition $u_i(x,0) = 0$). Substituting these expressions into Eq. (B.1), we get the recursion relations

$$\text{(B.5)} \qquad (\lambda + k_i)u_i^{\{\beta\lambda\}} = Q_i^{\{\beta\lambda\}}(u_j^{\{\gamma\delta\}}, A_{ik}^{a\{\alpha\rho\sigma\}}) + R_i^{\{\beta\lambda\}}(u_j^{\{\gamma\delta\}}, B_i^{\{\alpha\rho\sigma\}})$$

where $Q_i^{\{\beta\lambda\}}$ and $R_i^{\{\beta\lambda\}}$ are polynomials in their arguments with non-negative coefficients and which are linear and homogeneous in the $A_{ik}^{a\{\alpha\rho\sigma\}}$ and $B_i^{\{\alpha\rho\sigma\}}$ respectively. Furthermore, $Q_i^{\{\beta\lambda\}}$ and $R_i^{\{\beta\lambda\}}$ involve only those $u_j^{\{\gamma\delta\}}$ with $\delta < \lambda$. Thus one can also write (for $\lambda = 1, 2, \dots$)

$$\text{(B.6)} \qquad \begin{aligned} u_i^{\{\beta\lambda\}} &= Q_i^{\{\beta\lambda\}}\left(u_j^{\{\gamma\delta\}}, \frac{1}{\lambda + k_i}A_{ik}^{a\{\alpha\rho\sigma\}}\right) \\ &\quad + R_i^{\{\beta\lambda\}}\left(u_j^{\{\gamma\delta\}}, \frac{1}{\lambda + k_i}B_i^{\{\alpha\rho\sigma\}}\right) \end{aligned}$$

and derive, by successive substitutions to eliminate the $u_j^{\{\gamma\delta\}}$, the expressions

$$\text{(B.7)} \qquad \begin{aligned} u_i^{\{\beta\lambda\}} &= P_i^{\{\beta\lambda\}}\left(\frac{1}{\gamma + k_j}A_{jk}^{a\{\alpha\rho\sigma\}}, \frac{1}{\gamma + k_j}B_j^{\{\alpha\rho\sigma\}}\right), \\ \gamma &= 1, 2, \dots, \lambda, \end{aligned}$$

where $P_i^{\{\beta\lambda\}}$ are polynomials in the arguments $\frac{1}{\gamma + k_j}A_{jk}^{a\{\alpha\rho\sigma\}}$ and $\frac{1}{\gamma + k_j}B_j^{\{\alpha\rho\sigma\}}$ with non-negative coefficients. (Note that since some of the $\frac{1}{\gamma \mp k_l}$ may be negative, one cannot in general absorb these factors into the polynomial coefficients without disturbing their non-negativity).

Now let $P_i^{\{\beta\lambda\}}\left(\frac{K}{\gamma} A_{jk}^{a\{\alpha\rho\sigma\}}, \frac{K}{\gamma} B_j^{\{\alpha\rho\sigma\}}\right)$ denote the polynomials obtained from those in Eq. (B.7) by setting $k_i = 0$ and replacing $A_{jk}^{a\{\alpha\rho\sigma\}}$ and $B_j^{\{\alpha\rho\sigma\}}$ by $KA_{jk}^{a\{\alpha\rho\sigma\}}$ and $KB_j^{\{\alpha\rho\sigma\}}$ respectively, where K is a positive constant (to be chosen below). Since the factors $\frac{K}{\gamma}$ are strictly positive, one may absorb them into the coefficients and define

$$\mathring{P}_i^{\{\beta\lambda\}}\left(A_{jk}^{a\{\alpha\rho\sigma\}}, B_j^{\{\alpha\rho\sigma\}}\right) = P_i^{\{\beta\lambda\}}\left(\frac{K}{\gamma} A_{jk}^{a\{\alpha\rho\sigma\}}, \frac{K}{\gamma} B_j^{\{\alpha\rho\sigma\}}\right) \tag{B.8}$$

where the $\mathring{P}_i^{\{\beta\lambda\}}$ are polynomials in the indicated arguments with non-negative coefficients. Clearly the formal solution to the non-singular problem (B.2) is given by

$$u_i^{\{\beta\lambda\}} = \mathring{P}_i^{\{\beta\lambda\}}\left(A_{jk}^{a\{\alpha\rho\sigma\}}, B_j^{\{\alpha\rho\sigma\}}\right) \tag{B.9}$$

while the formal solution to the singular problem (B.1) is given by (B.7).

We now choose K such that

$$K \geqslant \max_{\gamma \in Z^+} \left(\left| \frac{\gamma}{\gamma + k_1} \right|, \ldots, \left| \frac{\gamma}{\gamma + k_N} \right| \right) \tag{B.10}$$

where $Z^+ = \{1, 2, \ldots\}$ This is always possible since the k_i are excluded from being negative integers. This choice ensures that

$$\begin{aligned} &P_i^{\{\beta\lambda\}}\left(\left| \frac{1}{\gamma + k_j} A_{jk}^{a\{\alpha\rho\sigma\}} \right|, \left| \frac{1}{\gamma + k_j} B_j^{\{\alpha\rho\sigma\}} \right| \right) \\ &\leqslant P_i^{\{\beta\lambda\}}\left(\left| \frac{K}{\gamma} A_{jk}^{a\{\alpha\rho\sigma\}} \right|, \left| \frac{K}{\gamma} B_j^{\{\alpha\rho\sigma\}} \right| \right) \\ &= \mathring{P}_i^{\{\beta\lambda\}}\left(\left| A_{jk}^{a\{\alpha\rho\sigma\}} \right|, \left| B_j^{\{\alpha\rho\sigma\}} \right| \right) . \end{aligned} \tag{B.11}$$

It follows that if $\{A', B'\}$ is a set of arguments which majorizes $\{A, B\}$ in the sense that

$$\begin{aligned} &\left| A_{jk}^{a\{\alpha\rho\sigma\}} \right| \leqslant A_{jk}^{\prime a\{\alpha\rho\sigma\}} , \\ &\left| B_j^{\{\alpha\rho\sigma\}} \right| \leqslant B_j^{\prime\{\alpha\rho\sigma\}} \end{aligned} \tag{B.12}$$

then we have (writing (A_{jk}, B_j) for $(A_{jk}^{a\{\alpha\rho\sigma\}}, B_j^{\{\alpha\rho\sigma\}})$ to simplify the notation)

$$\begin{aligned}
&\left|P_i^{\{\beta\lambda\}}\left(\frac{1}{\gamma+k_j}A_{jk},\frac{1}{\gamma+k_j}B_j\right)\right| \\
&\leqslant P_i^{\{\beta\lambda\}}\left(\left|\frac{1}{\gamma+k_j}A_{jk}\right|,\left|\frac{1}{\gamma+k_j}B_j\right|\right) \\
&\leqslant \mathring{P}_i^{\{\beta\lambda\}}(|A_{jk}|,|B_j|) \\
&\leqslant \mathring{P}_i^{\{\beta\lambda\}}(A'_{jk},B'_j).
\end{aligned} \tag{B.13}$$

Thus a majorant for the non-singular problem is also a majorant for the singular one.

Since the standard Cauchy-Kowalewski theorem [14] provides a majorant for the non-singular problem, we thus get a majorant for the singular problem and thus a proof that the formal series solution to the singular problem converges absolutely. This proves

Theorem (1): Equation (B.1) with the initial condition $u_i(x, 0) = 0$ has a unique analytic solution on a neighborhood of the origin.

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[14] See, for example, Chap. 3 of Ref. [13].

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